Mathematics Framework
Chapter 1: Introduction to the 2021 Mathematics Framework

First Field Review Draft

Note to reader: The use of the non-binary, singular pronouns *they*, *them*, *their*, *theirs*, *themself*, and *themselves* in this framework is intentional.

Introduction

> *A society without mathematical affection is like a city without concerts, parks, or museums. To miss out on mathematics is to live without an opportunity to play with beautiful ideas and see the world in a new light.*
>
> *—Francis Su (2020)*

Welcome to the 2021 *Mathematics Framework for California Public Schools, Kindergarten Through Grade Twelve* (*Math Framework*). This framework serves as a guide to implementing the California Common Core State Standards for Mathematics (CA CCSSM or the Standards). Built upon underlying and updated principles of *focus*, *coherence*, and *rigor*, the Standards hold the promise of enabling all California students to become powerful users of mathematics in order to better understand and positively impact the world—in their careers, in college, and in civic life.

Mathematics as a Gatekeeper or a Launchpad?

> *Be careful how you interpret the world: It is like that.*
>
> *—Erich Heller (1952)*

Mathematics provides a set of lenses that provide important ways to understand many situations and ideas. The ability to use these mathematical lenses flexibly and accurately enables the people of California to apply mathematical understandings to influence their communities and the larger world in important ways. Mathematics continues to play a role in how we conceive of our careers, evidence-based civic discourse and policy-making, and the examination of assumptions and principles underlying action. All students are capable of making these contributions and achieving these abilities at the highest levels. As a guide to implementing the Standards, this framework outlines innovative mathematical learning experiences with the potential to help all California students.

To develop learning that can lead to mathematical power for all California students, the framework has much to correct; the subject and community of mathematics has a history of exclusion and filtering, rather than inclusion and welcoming. There persists a mentality that some people are “bad in math” (or otherwise do not belong), and this mentality pervades many sources and at many levels. Girls and Black and Brown children, notably, represent groups that more often receive messages that they are not capable of high-level mathematics, compared to their White and male counterparts (Shah & Leonardo, 2017). As early as preschool and kindergarten, research and policy documents use deficit-oriented labels to describe Black and Latinx and low-income children's mathematical learning and position them as already behind their white and middle-class peers (NCSM & TODOS, 2016). These signifiers exacerbate and are exacerbated by acceleration programs that stratify mathematics pathways for students as early as sixth grade.

Students internalize these messages to such a degree that undoing a self-identity that is "bad at math" to one that "loves math" is rare. Before students have opportunities to excel in mathematics, many often self-select out of mathematics because they see no relevance for their learning, and no longer recognize the inherent value or purpose in learning mathematics. The fixed mindset about mathematics ability reflected in these beliefs helps to explain the exclusionary role that mathematics plays in students' opportunities, and leads to widespread inequities in the discipline of mathematics. Some of these include:

- Students who are perceived as "weak" in mathematics are often informally tracked before grade seven in ways that severely limit their experiences with and approaches to mathematics (Butler, 2008) and their future options (Parker et al, 2014). See also Chapter 8.
- Students who do not quickly and accurately perform rote procedures get discouraged and decide not to persist in mathematically-oriented studies.
- Students who are learning the English language are deemed incapable of handling, and denied access to, grade-level authentic mathematics.
- Students with learning differences that affect performance on computational tasks are denied access to richer mathematics, even when the learning differences might not affect other mathematical domains (Lambert, 2018).
- Students who are tracked into lower mathematics courses in middle and high school can be denied entry into prestigious colleges.

Many factors contribute to mathematics exclusion. As one example, consider a system described in more detail in Chapters 7 (Grades 6–8) and 8 (Grades 9–12): Though many high schools offer integrated mathematics, high school mathematics courses are often structured in such a way (e.g., algebra-geometry-algebra 2- precalculus) calculus is considered the main course for Science, Technology, Engineering, Arts, and Mathematics (STEAM)-oriented students, and is only available to students who are considered “advanced” in middle school—that is, taking algebra in eighth grade. In order to reach algebra in grade eight, students must cover all of middle grades math in just two years (or else skip some foundational material). This means that many school systems are organized in ways that ultimately decide which students are likely to go into STEAM pathways when they begin sixth grade. This reality leads to considerable racial- and gender-based inequities and filters out the majority of students out of a STEAM pathway (Joseph, Hailu, Boston, 2017). Moreover, English learners have disproportionately less access, are placed more often in remedial classes and are steered away from STEAM courses and pathways (National Academies of Sciences, Engineering, and Medicine, 2018). High school mathematics courses such as data science should exist as a viable option whether students consider STEAM or non-STEAM career options.

Considering that many competitive colleges and universities (those that accept less than 25 percent of applicants) hold calculus as an unstated requirement, the inequitable pathway becomes even more problematic. Many students remain unaware that their status at the end of fifth grade can determine their ability to attend a top university; if they are not in the advanced mathematics track and on a pathway to calculus in each of the subsequent six years of school, they will not meet this unstated admission requirement. This mathematics pathway system, typical of many school districts, counters the evidence that shows all fifth graders are capable of eventually learning calculus, or other high-level courses, *when provided appropriate messaging, teaching, and support*. The system of providing only some students pathways to calculus, or statistics, data science or other high-level courses has resulted in the denial of opportunities too many potential STEAM students—especially Latinx and African American students. At the same time, arbitrary or irrelevant mathematics hurdles block too many students from pursuing non-STEAM careers. Mathematics education must support students whether they wish to pursue STEAM disciplines or any other promising major that prepares them for careers in other fields, like law, politics, design, and the media. Mathematics also needs to be relevant for students who pursue careers directly after high school, without attending college (Daro & Asturias, 2019). Schooling practices that lead to such race- and gender-based disparities can lead to legal liabilities for districts and schools (Lawyers' Committee for Civil Rights of the San Francisco Bay Area, 2013). A fuller discussion of one example is included in Chapter 8. The middle- and high-school chapters (Chapters 7 and 8), and the data science chapter (Chapter 5) outline an approach that enables all students to move to calculus, data science, statistics, or other high level courses, with grade level courses, 6, 7, and 8 in middle school. The new provision of a data science high school course, open to all students(not only those considered "advanced" in middle school), that can serve as a replacement for algebra 2, has the potential to open STEAM pathways to diverse groups of students, both through its engaging content and its openness to all students—as described further in Chapter 5, and Chapters 7 and 8.

Mathematics education can also create the levels of understanding that can launch student action, both locally and globally. While every level of schooling must focus on providing access to mathematical power for *all* students, changing the high-school level mathematics remains a critical component to opening mathematics doorways for all students. In *Catalyzing Change in Middle School Mathematics,* NCTM suggests that the purpose of school mathematics expand to include the development of positive mathematical identities and a strong sense of agency (see Aguirre, Mayfield-Ingram, & Martin, 2013). NCTM further urges educators to focus on dismantling structural obstacles that stand in the way of rich mathematical experiences for all students, and organize middle-school mathematics along a common, shared pathway grounded in the use of mathematical practices and processes that support mathematical understanding. Pathways that provide access to higher-level mathematics from a typical grade nine course are described in Chapter 8. In local educational agencies (LEAs) where high school administrators commit to such pathways and vow to support communities of teachers and students in succeeding in grade-level appropriate mathematics, middle school pathways can avoid compressing or skipping important mathematical courses that can speed students through fundamental content. Nor will teachers need to track students into different pathways. More fundamentally, all stakeholders need to work to shift the definition of mathematics success away from acceleration, and focus on depth of learning.

Learning Mathematics: for All

Introduction

Students learn best when they are actively engaged in questioning, struggling, problem solving, reasoning, communicating, making connections, and explaining. The research is overwhelmingly clear that powerful mathematics classrooms thrive when students feel a sense of agency (a willingness to engage in the discipline, based in a belief in progress through engagement) and an understanding that the intellectual authority in mathematics rests in mathematical reasoning itself (in other words, that mathematics makes sense) (Boaler, 2019 a, b; Boaler, Cordero & Dieckmann, 2019; Anderson, Boaler & Dieckmann, 2018; Schoenfeld, 2014). These factors support students as they develop their own identities as powerful mathematics learners and users. Further, active-learning experiences enable students to engage in a full range of mathematical activities—exploring, noticing, questioning, solving, justifying, explaining, representing and analyzing—making clear that mathematics represents far more than calculating.

Research is also clear that *all* students are capable of becoming powerful mathematics learners and users (Boaler, 2019a, c). This notion runs counter to many students' ideas about school mathematics. Most adults can recall times when they received messages during their school or college years that they were not cut out for mathematics-based fields. The race-, class-, and gender-based differences in those who pursue more advanced mathematics make it clear that messages students receive about who belongs in mathematics are biased along racial, socioeconomic status, language, and gender lines, a fact that has led to considerable inequities in mathematics.

In 2015, Sarah-Jane Leslie, Andrei Cimpian, and colleagues interviewed university professors in different subject areas to gauge student perceptions of educational “gifts”—the concept that people need a special ability to be successful in a particular field. The results were staggering; the more prevalent the idea, in any academic field, the fewer women and people of color participating in that field. This outcome held across all thirty subjects in the study. More mathematics professors believed that students needed a gift than any other professor of STEAM content. The study highlights the subtle ways that students are dissuaded from continuing in mathematics and underscores the important role mathematics teachers play in communicating messages that mathematics success is only achievable for select students. This pervasive belief more often influences women and people of color to conclude they will not find success in classes or studies that rely on knowledge of mathematics.

Negative messages, either explicit (“I think you’d be happier if you didn’t take that hard mathematics class”) or implicit (“I’m just not a math person”), both imply that only some people can succeed. Perceptions can also be personal (“Math just doesn’t seem to be your strength”) or general (“This test isn’t showing me that these students have what it takes in math.” My other class aced this test.”). And they can also be linked to labels (“low kids,” “bubble kids,” “slow kids”) that lead to a differentiated and unjust mathematics education for students.

A fundamental aim of this framework is to respond issues of inequity in mathematics learning; equity influences all aspects of this document. Some overarching principles that guide work towards equity in mathematics include the following:

- Access to an engaging and humanizing education—a socio-cultural, human endeavor—is a universal right, central among civil rights.
- All students deserve powerful mathematics; we reject ideas of natural gifts and talents (Cimpian et al, 2015; Boaler, 2019) and the “cult of the genius” (Ellenberg, 2015).
- The belief that “I treat everyone the same” is insufficient: Active efforts in mathematics teaching are required in order to counter the cultural forces that have led to and continue to perpetuate current inequities (Langer-Osuna, 2011).
- Student engagement must be a design goal of mathematics curriculum design, co-equal with content goals.

- Mathematics pathways must open mathematics to all students, eliminating option-limiting tracking.
- Students' cultural backgrounds, experiences, and language are resources for learning mathematics (González, Moll, & Amanti, 2006; Turner & Celedón-Pattichis, 2011; Moschkovich, 2013).
- All students, regardless of background, language of origin, differences, or foundational knowledge are capable and deserving of depth of understanding and engagement in rich mathematics tasks.

Rejecting Fixed Ideas about Students

> *Hard work and persistence is more important for success in mathematics than natural ability. Actually, I would give this advice to anyone working in any field, but it's especially important in mathematics and physics where the traditional view was that natural ability was the primary factor in success."*
>
> *—Maria Klawe, Mathematician, Harvey Mudd President (in Williams, 2018)*

Fixed notions about student ability, such as ideas of "giftedness," have led to considerable inequities in mathematics education. Particularly damaging is the idea of the "math brain"—that people are born with a brain that is suited (or not) for math. Technologies that have emerged in the last few decades have allowed researchers to understand the mind and brain and completely challenged this idea. With current technology, scientists can study learning in mathematics through brain activity; they can look at growth and degeneration and see the impact of different emotional conditions on brain activity. This work has shown—resoundingly—that all people possess the capacity to learn mathematics to very high levels. Multiple studies have shown the incredible capacity of brains to grow and change within a short period of time (Huber et al, 2018; Luculano et al, 2015; Abiola & Dhindsa, 2011; Maguire, Woollett, & Spiers, 2006; Woollett & Maguire, 2011). Learning allows brains to form, strengthen, or connect brain pathways in a process of almost constant change and adaptation (Doidge, 2007; Boaler, 2019a). An important goal of this framework is to replace ideas of innate mathematics “talent” and “giftedness” with the recognition that every student is on a growth pathway. There is no cutoff determining when one child is “gifted” and another is not.

The neuroscientific evidence that shows the potential of all students to reach high levels in mathematics is the evidence base that supports the importance of mindset messages. Stanford University psychologist Carol Dweck and her colleagues have conducted research studies in different subjects and fields for decades showing that people’s beliefs about personal potential changes the ways their brains operate and influences what they achieve. One of the important studies Dweck and her colleagues conducted took place in mathematics classes at Columbia University (Carr et al., 2012), where researchers found that young women received messaging that they did not belong in the discipline. When students with a fixed mindset heard the message that math was not for women, they dropped out. Those with a growth mindset, however, protected by the belief that anyone can learn anything, ultimately rejected the stereotype and persisted. Dweck and her colleagues have shown, through multiple studies, that students with a growth mindset achieve at higher levels in mathematics, and that when students change their mindsets, from fixed to growth, their mathematics achievement increases (Blackwell, Trzesniewski & Dweck, 2007; Boaler, 2019).

Another idea related to the “math brain” that teachers should challenge comes from social comparison. Students often believe that brains must be fixed, because some people appear to get ideas faster and to be naturally good at certain subjects. What these students do not realize is that brains grow and change every day. Each moment is an opportunity for brain growth and development and some students have developed stronger pathways on a different timeline. Teachers should strive to reinforce the idea that all students can develop those pathways at any time if they take the right approach to learning.

It is important for teachers to share the science of brain growth and clarifying the idea that, although students are all unique, anyone can learn the content that is being taught, and productive learning is in part due to their thinking. This understanding can be particularly effective at the beginning of the school year or math course. Students may find the message liberating, and allow it to override any prevailing messaging from teachers that success in math can only be achieved by a few students. When students learn about brain growth and mindset, they realize something critically important—no matter where they are in their learning, they can improve and eventually excel (Blackwell, Trzesniewski & Dweck, 2007). Teachers should also underscore the importance and value of times of struggle. This understanding comes, in part, from psychologist Jason Moser and his colleagues, who found that when adults were taking tests, they experienced more brain growth and activity when they made mistakes than when they achieved correct answers (Moser, et al, 2011). This fits into a range of neuroscientific work showing that times of struggle are productive for brains as they are the times that pathways are developing and strengthening. The importance of struggle has been shown through both brain-based and behavior-based studies. Daniel Coyle (2009), for example, studied the highest achieving people in different fields of work and found a characteristic shared by these achievers was a willingness to struggle—to work "at the edge of their understanding," to make mistakes, correct them, move on, and create more. This, he found, was the optimal approach to accelerate learning. This evidence becomes particularly important when we consider that students often struggle in math class, decide they do not have a "math brain," and give up. It is important for teachers to share the research on the benefits of and encourage students to persevere when it seems easier to give up. Various videos for sharing messages about mindset and the value of struggle are provided at https://www.youcubed.org/resource/mindset-boosting-videos/.

The significance of changing the ways teachers, parents, and others, consider students with different learning needs—because they are higher achieving, learning English, or have learning differences, is considered below:

Linguistically and Culturally Diverse Learners

> *In mathematics and mathematics education, the important step is to accept other ways of knowing and other forms of mathematical activity. The history of mathematics, when we focus on the dynamics of cultural encounters, is,*

effectively, mankind’s worldwide, transcultural endeavor in the search for survival and transcendence. Only a limited and biased view of history tells us that this search is the privilege of a specific culture.

Ubiratan D’Ambrosio, *Culturally Responsive Mathematics Education* (2009)

Humans are all hardwired to learn mathematics (Devlin 2001). Regardless of culture or origin, humans can already distinguish between quantities during infancy (Lipton & Spelke 2003). Despite this, schooling experiences can either support individuals’ natural mathematical curiosities and competencies or diminish them. In fact, NAEP data continue to show the influence of schooling on English learners, who experience greater mathematical disconnection. These trends have typically been framed and interpreted in deficit terms. Yet the universal capacity for mathematics points to a different interpretation; mathematics education in the United States has not been designed to meet the needs of many students in our culturally and linguistically diverse society (Gutierrez, 2008).

Mathematics education in the United States was initially structured for a narrow purpose: to prepare privileged, young, white men for entrance into elite colleges. Harvard University chose to make arithmetic a college requirement based on the belief at the time that the mind was a muscle that could be trained through exercise and drills, just like the body. Those in positions of authority designed secondary schools to offer the mathematics courses that Harvard required for entrance: arithmetic, algebra, geometry, and advanced topics (Furr, 1996). While instruction has shifted toward learning with understanding, and the field increasingly attends to issues of equity and access, mathematics education still largely recreates this rigid and rote approach to mathematics teaching and learning; achievement in mathematics often reflects these original, narrow purposes. These foundations continue to limit the experience of mathematics as relevant, meaningful, and engaging and obscure many student competencies that could otherwise be drawn upon to support making sense of mathematics. This is particularly true for linguistically and culturally diverse learners of English, whose competencies have long been obscured through deficit frameworks and narrow conceptions of mathematical competence. This framework includes the fact that these students are “linguistically and culturally diverse,” and draws from The Coalition for English Learner Equity’s (CELE) website,

> to describe a heterogeneous group of learners that includes students learning in Dual Language contexts, students who are multilingual, and students who are bureaucratically labeled as English learners. These are students for whom language is a reason for their minoritization due to systemic racism, but also for whom language, culture, and literacy are their greatest assets" (http://elequity.org, footnote 3).

Far from being a cultural-free environment, mathematics classrooms function as communities of learners. Educators and administrators continue to gain key insights into how teachers and students create mathematics learning communities that are engaging, inclusive, and rigorous. Culturally responsive mathematics education, for example, emphasizes active, collaborative communities of learners engaged in mathematical explorations through meaningful and personally-relevant social contexts (Powell, Mukhopadhyay, Nelson-Barber, & Greer, 2009). And studies on language and mathematics education highlight the importance of centering students’ cultural and linguistic competencies and identities in defining particular communities of learners (Moschkovich, 2009; Turner, et al, 2013).

Students with Learning Differences

The evidence that all students have the potential to reach high levels is particularly important for students diagnosed with special needs, many of whom are set on low-level pathways, even as research is showing the capacity of all brains to rewire and change (Boaler & LaMar, 2019). Across the United States, approximately 8.4 percent of students are diagnosed as having a special education need. The vast majority of those—72 percent—are diagnosed as having mild to moderate needs, including learning differences such as dyslexia, dyscalculia, and auditory processing disorder. Inequities persist in special education just as they do in most other aspects of schooling. For example, males and students of color are more frequently classified as special education students than females and white mainstream students. Nearly twice as many males as females are classified as students with learning differences. The group most likely to be classified as “mentally retarded” or “learning disabled” are boys of color. Black students with learning differences are four times more likely than their white counterparts to be educated in correctional facilities. Although the field of special education has traditionally referred to students with special needs as being “learning disabled,” documenting various “disabilities” that require attention, we prefer the term “learning differences.” This gives an asset, rather than deficit, framing, acknowledging that students may have a need for learning support but this does not mean they should be viewed as limited or “disabled.”

Further, new and promising research is showing that students can develop the brain pathways they need and lose the need for learning assistance. In one study researchers gave a learning intervention to 24 children ranging from seven to twelve years old who were either clinically diagnosed with dyslexia or recorded as having significant reading difficulties (Huber et al, 2018). These children were given an intensive eight-week long reading intervention program where they participated in one-on-one training sessions for four hours a day, five days a week. The researchers found large-scale changes in brain growth for the participants. Furthermore, this brain growth was correlated with a significant improvement in reading skills. By the end of the intervention program, the average reading achievement score for the intervention group was within the range of scores for typical readers (Huber et al, 2018). A different intervention studying mathematics, conducted by neuroscientist Teresa Luculano and her colleagues in Stanford’s school of medicine, was similarly promising (Luculano et al, 2015). The researchers brought in children from two groups—one group had been diagnosed as having mathematics “learning disabilities” and the other consisted of regular performers. The researchers examined scans of the children’s brains taken when they were working on mathematics. They found actual brain differences—the students identified as having disabilities had more brain regions illuminated when they worked on a mathematics problem. The researchers provided one-to-one tutoring for both sets of students—those who were regular performers and those identified as having a mathematics learning difference. The tutoring, which included eight weeks of 40–50 minute sessions per day, focused on strengthening student understanding of relationships between and within operations. At the end of the eight weeks of tutoring, not only did both sets of students have the same achievement; they also activated the same brain areas. Both of these studies show that in a short period of time with careful teaching, brains can be changed and rewired. Such studies should remind us that all students are on a growth journey. The dichotomous thinking that fills schools—with decisions that some students are "smart" or capable of high-level work, while others have "learning disabilities"—does not appear justified when considering the latest work in brain growth from neuroscience and elsewhere, and has created significant inequalities in mathematics. The idea of student inadequacy has often been based on a mathematics approach that is narrow and speed based. When mathematics is made multi-dimensional (see below), and depth is valued over speed, different students are able to access ideas and connect with the mathematics. The guidelines in Universal Design for Learning (or UDL) show the importance of teaching in a more multi-dimensional way—sharing ideas and valuing student input with multiple forms of engagement, representation and expression (https://udlguidelines.cast.org/). Adopting the perspective that learning differences represent strengths and more multidimensional teaching can allow all students to be successful.

High Achieving Students

In previous versions of this framework, students who have shown higher achievement than their peers have been given fixed labels of "giftedness" and taught differently. Such labelling has often led to fragility among students, who fear times of struggle in case they lose the label (see, for example: https://www.youcubed.org/rethinking-giftedness-film/), as well as significant racial divisions. In California in the years 2004–2014, 32 percent of Asian American students were in gifted programs compared with 8 percent of White students, 4 percent of Black students, and 3 percent of Latinx students (https://nces.ed.gov/programs/digest/d17/tables/dt17_204.80.asp).

While many districts have moved away from such labelling and the resulting differential treatment, students who achieve at high levels can still suffer from a faster paced (and often shallower) mathematics experience—one that does not lead to depth of understanding or appreciation of the content. The legacy of mathematics education as both "mental training" and as a sort-of access code for higher education have undercut meaningful learning, reducing mathematics to a high-stakes performance for the college-bound student, and as an arbitrary hurdle for all others. Even for the highest-achieving students, pressures to use mathematics courses as social capital for advancement can often undercut efforts to promote learning with understanding. This often results in what some deem a “rush to calculus,” which has not helped students. Bressoud (2017) studied the mathematics pathways of students moving from calculus to college. He found that out of the 800,000 students who take calculus in high school, roughly 250,000 or 31.25 percent of students move ‘backwards’ and take precalculus, college algebra, or remedial mathematics. Roughly 150,000 students take other courses such as Business Calculus, Statistics, or no mathematics course at all. Another 250,000, retake Calculus 1 and of these students about 60 percent of them earn an A or B and 40 percent earn a C or lower. Only 150,000 or 19 percent of students go on to Calculus II. This signals that the approach that is so prevalent in schools—of rushing students to calculus, without depth of understanding—is not helping their long term mathematics preparation. This has led the Mathematical Association of America (MAA) and the National Council of Teachers of Mathematics (NCTM) to issue the following joint statement:

> Although calculus can play an important role in secondary school, the ultimate goal of the K–12 mathematics curriculum should not be to get students into and through a course in calculus by twelfth grade but to have established the mathematical foundation that will enable students to pursue whatever course of study interests them when they get to college. The college curriculum should offer students an experience that is new and engaging, broadening their understanding of the world of mathematics while strengthening their mastery of tools that they will need if they choose to pursue a mathematically intensive discipline. (http://launchings.blogspot.com/2012/04/maanctm-joint-position-on-calculus.html)

Other studies give insights into the reasons that students do not do well when rushed through mathematics courses, particularly in the field of de-tracking. Burris, Heubert & Levin (2006) followed students through middle schools in the district of New York. In the first three years, the students were in regular or advanced classes, in the following three years all students took the same mathematics classes comprised of advanced content. In their longitudinal study the researchers found that when all students learned together the students achieved more, took more advanced courses in high school, and passed state exams a year earlier, with achievement advantages across the achievement range, including the highest achievers (Burris, Heubert & Levin, 2006). In a study with similar findings, conducted in the California Bay Area, eight school districts de-tracked middle school mathematics and gave professional development to the teachers. In 2014 63 percent of students where in advanced classes, in 2015 only 12 percent were in advanced classes and everyone else was taking Common Core math 8. The overall achievement of the students after the de-tracking significantly increased. The cohort of students who were in eighth-grade mathematics in 2015 were 15 months ahead of the previous cohort of students who were mainly in advanced classes (MAC & CAASPP 2015). Educators in the San Francisco Unified School District found similar benefits when they delayed any students taking advanced classes in mathematics until after tenth grade and moved the algebra course from eighth to ninth grade. After making this change the proportion of students failing algebra fell from 40 percent to eight percent, and the proportion of students taking advanced classes rose to a third of the students, more than any other number in the history of the district (Boaler et al, 2018).

One of the reasons that students are often limited when in tracked groups is the questions given to the students are narrow, which precludes access for some students and stops higher achievers from taking the work to higher levels. When schools de-track, and teachers move to giving differentiated work or more open mathematics questions that can reflect different levels, students of all achievement levels benefit. All this evidence supports the belief that students are best served working on mathematics at a reasonable pace—not rushing coursework means that high achievers can take work to deeper levels rather than speed ahead with superficial understanding of content, and learn to appreciate the beauty of mathematics and the connections between mathematical areas. All students can take Common Core-aligned mathematics 6, 7, and 8 in middle school and still take calculus, data science, statistics, or other high-level courses in high school.

Multi-dimensional Mathematics

A third meaningful result from studies of the brain is the importance of brain connections. Vinod Menon (2015) and a team of researchers at Stanford University have studied the interacting networks in the brain, particularly focusing on the ways the brain works when it is solving problems—including mathematics problems. They found that even when people are engaged with a simple arithmetic question, five different areas of the brain are involved, two of which are visual pathways. The dorsal visual pathway is the main brain region for representing quantity.

Menon and other neuroscientists also found that communication between the different brain areas enhances learning and performance. Researchers Joonkoo Park and Elizabeth Brannon (2013) reported that different areas of the brain were involved when people worked with symbols, such as numerals, than when they worked with visual and spatial information, such as an array of dots. The researchers also found that mathematics learning and performance were optimized when these two areas of the brain were communicating with each other. Learning mathematical ideas comes not only through numbers, but also through words, visuals, models, algorithms, multiple representations, tables, and graphs; from moving and touching; and from other representations. But when learning reflects the use of two or more of these means and the different areas of the brain responsible for each communicate with each other, the learning experience improves.

For this reason, this framework highlights examples that are multi-dimensional, with mathematical experiences that are visual, physical, numerical, and more. These approaches align with the principles of Universal Design for Learning (UDL), a framework designed to make learning more accessible, that helps all students. Visual and physical representations of mathematics are not only for young children, nor are they merely a prelude to abstraction or higher-level mathematics (Boaler et al, 2016). Some of the most important high-level mathematical work and thinking—such as the work of Fields medal winner Maryam Mirzakhani—is visual.

The different areas of neuroscientific research with evidence showing the potential of brains to grow and change, the importance of times of struggle, and the value in engaging with mathematics in multi-dimensional ways, should be shared with students. When messages such as these were shown in a free online class offered through a randomized controlled trial, students significantly increased their mathematics engagement in class and improved later achievement (Boaler et al, 2018). This information is shared through freely available lessons and videos on https://youcubed.org.

Mathematics: Tools for Making Sense

Without mathematics, there's nothing you can do. Everything around you is mathematics. Everything around you is numbers.

—Shakuntala Devi, Author & "Human Calculator"

Mathematics grows out of curiosity about the world. Humans are born with an intuitive sense of numerical magnitude (Feigenson, Dehaene, & Spelke 2004), and this intuitive sense develops in early life into knowledge of number words, numerals, and the quantities they represent.

Give babies a set of blocks, and they will build and order them, fascinated by the ways the edges line up. Children will look up at the sky and be delighted by the V formations in which birds fly. Count a set of objects with a young child, move the objects and count them again, and they will be enchanted by the fact they still have the same number. Human minds want to see and understand patterns (Devlin, 2006). But the joy and fascination young children experience with mathematics is quickly replaced by dread and dislike when mathematics is introduced as a dry set of methods they think they just have to accept and remember.

Young students' work in mathematics is firmly rooted in their experiences in the world (Piaget and Cook, 1952). Numbers name quantities of objects or measurements such as time and distance, and operations such as addition and subtraction are represented by manipulations of such objects or measurements. Soon, the whole numbers themselves become a context that is concrete enough for students to grow curious about and to reason within—with real-world and visual representations always available to support reasoning.

Students who use mathematics powerfully can maintain this connection between mathematical ideas and meaningful contexts. Historically, too many students lose the connection at some point between primary grades and graduation from high school. The resulting experience creates students who see mathematics as an exercise in memorized procedures that match different problem types.

The broad themes of this framework encompass four points:

1. The work of students as mathematicians requires them to engage with content and the SMPs through both oral and written language;
2. Teachers need to attend to students' development of mathematical content, SMPs, and language;
3. Mathematics content is best approached through a focus on big ideas, investigation, and connections across content; and
4. Broadening mathematical competence through teaching and assessment mathematics creates more inclusivity grounded in students' lived experiences.

This framework adopts the implicit understanding that all students are capable of accessing and mastering school mathematics in the ways envisioned in California Common Core Standards for Mathematics (CA CCSSM). "Mastering" means becoming inclined and able to consider novel situations (arising either within or outside mathematics) through a variety of appropriate mathematical tools, using those tools to understand the situation and, when desired, to exert their own power to affect the situation. Thus, mathematical power is not reserved for a few, but available to all.

Translating this potential into reality requires a school mathematics system built to achieve this purpose. Current structures often reinforce existing factors that allow access for some while telling others they don't belong; structures must instead challenge those factors by providing relevant, authentic mathematical experiences that make it clear to all students that mathematics is a powerful tool for making sense of and affecting their worlds. This will be an important contribution to equitable outcomes.

Audience

The *Math Framework* is intended to serve many different audiences, each of whom contribute to the shared mission of helping all students become powerful users of mathematics as envisioned in the CA CCSSM. First and foremost, the *Math Framework* is written for teachers and those educators who have the most direct relationship with students around their developing proficiency in mathematics. As in every academic subject, developing powerful thinking requires contributions from many; and so this framework is also directed to:

- parents and caretakers of K–12 students who represent crucial partners in supporting their students' mathematical success;
- curricular materials designers and authors whose products help teachers to implement the Standards through engaging, authentic classrooms;
- educators leading pre-service and teacher preparation programs whose students face a daunting but exciting challenge of preparing to engage students in meaningful, coherent mathematics;
- in-service professional learning providers who can help teachers navigate deep mathematical and pedagogical questions as they strive to create coherent K–12 mathematical journeys for their students;
- instructional coaches and other key allies supporting teachers to improve students' experiences of mathematics;
- site, district, and county administrators who want to support improvement in mathematics experiences for their students;
- college and university instructors of California high school graduates who wish to use the framework in concert with the Standards to understand the types of knowledge, skills, and mindsets about mathematics that they can expect of incoming students;
- educators focused on other disciplines so that they can see opportunities for supporting their discipline-specific instructional goals while simultaneously reinforcing relevant mathematics concepts and skills; and

- assessment writers who create curriculum, state, and national tests that signal which content is important and the determine ways students should engage in the content.

Updating Coherence, Focus, and Rigor

The CA CCSSM were adopted by the State Board of Education in 2010 and modified in 2013. Over a decade of experiences have made evident the kinds of challenges the Standards posed for teachers, administrators, curriculum developers, professional learning providers, and others. When the Standards and the subsequent framework were each adopted, they both reflected an approach based on identifying major and minor standards—a recognition that it can be difficult for teachers to address all standards while maintaining a rich and deep learning experience for all students. This approach essentially eliminated key areas of content (such as data literacy). This framework reflects a revised approach, one that advocates for publishers and teachers avoiding the process of organizing around the detailed content standards, and instead establishing mathematics that reflect bigger ideas—those that connect many different standards in a more coherent whole. The *Math Framework* responds to challenges posed by each of the underlying principles.

Terms

Big Idea: Big ideas in math are central to the learning of mathematics, link numerous math understandings into a coherent whole, and provide focal points for students' investigations.

Drivers of Investigation (DIs): unifying reasons that both elicit curiosity and provide the motivation for deeply engaging with authentic mathematics (see end of this chapter)

Content Connections (CCs): content themes that provide mathematical coherence through the grades (see end of this chapter)

Authentic: An authentic problem, activity, or context is one in which students investigate or struggle with situations or questions about which they actually wonder. Lesson design should be built to elicit that wondering. In contrast, an activity is *inauthentic* if students recognize it as a straightforward practice of recently-learned techniques or procedures, including the repackaging of standard exercises in forced "real-world" contexts. Mathematical patterns and puzzles can be more authentic than such real-world settings.

Necessitate: An activity or task *necessitates* a mathematical idea or strategy if the attempt to understand the situation or task creates for students a need to understand or use the mathematical idea or strategy.

Coherence

I like crossing the imaginary boundaries people set up between different fields—it's very refreshing. There are lots of tools, and you don't know which one would work. It's about being optimistic and trying to connect things.

—Maryam Mirzakhani, Mathematician, 2014 Fields Medalist

Despite their differences and unique complexities, the Standards for Mathematical Practice (SMPs) and the Standards are intended to be equally important in planning, curriculum, and instruction (CA CCSSM [2013], p. 3). The content standards, however, are far more detailed at each grade level, and are more familiar to most educators; as a result, the content standards continue to provide the organizing structure for most curriculum and instruction. Because the content standards are more granular, curriculum developers and teachers find it easy when designing lessons to begin with one or two content standards and choose tasks and activities which develop that standard. Too often, this reinforces the concept as an isolated idea.

Because the Standards were then new to California educators (and to curriculum writers), the 2013 California *Mathematics Framework* was comprehensive in its treatment of the content standards; it included descriptions and examples throughout the framework for most. In the intervening years, many more examples, exemplars, and models of sample tasks representing illustrations of the mastery intended by each standard have emerged. Thus, the need is different in 2021: California teachers and students need mathematics experiences that provide access to the coherent body of understanding and strategies of the discipline.

Instructional materials should primarily involve tasks that invite students to make sense of these big ideas, elicit wondering in authentic contexts, and necessitate mathematical investigation. Big ideas in mathematics are central to the learning of mathematics, link numerous mathematical understandings into a coherent whole, and provide focal points for students' investigations. The value of focusing on big ideas for teachers, and their students, cannot be overstated. Voices in the field emphasize this: "When teachers work on identifying and discussing big ideas, they become attuned to the mathematics that is most important and that they may see in tasks, they also develop a greater appreciation of the connections that run between tasks and ideas" (Boaler, J., Munson, J., Williams, C., 2018). In each grade band section, the description focuses on several big ideas that have great impact on students' conceptual understanding of numbers, and which are connected to multiple elements of the content standards.

> Mathematical notation no more is mathematics than musical notation is music. A page of sheet music represents a piece of music, but the notation and the music are not the same; the music itself happens when the notes on the page are sung or performed on a musical instrument. It is in its performance that the music comes alive; it exists not on the page but in our minds. The same is true for mathematics.
>
> —Keith Devlin (2001)

An authentic activity or problem is one in which students investigate or struggle with situations or questions about which they actually wonder. Lesson design should be built to elicit that wondering.

This framework sets out these organizing ideas to provide *coherence* and to help teachers avoid losing the forest for the trees. That is, discrete content standard mastery does not necessarily assemble in students' minds into a coherent big-picture view of mathematics.

This framework's response to the challenge posed by the principle of coherence are: focusing on big ideas, both as Drivers of Investigation (the reasons why we do mathematics, see section below), and Content Connections (both within and across domains, see section below); progressions of learning across grades (thus, grade-band chapters rather than individual grade chapters); and relevance to students' lives. Principles guiding grade-band chapters include

- design from a smaller set of big ideas, spanning TK–12 in the forms of Drivers of Investigation (DIs) and Content Connections (CCs), within each grade band (see below);
- a preponderance of student time spent on authentic problems through the lenses of DIs and CCs (see below) that engage multiple content and practice standards situated within one or more big ideas;
- a focus on connections: between students' lives and mathematical ideas and strategies; and between different mathematical ideas; and
- constant attention to opportunities for students to bring other aspects of their lives into the math classroom: How does this mathematical way of looking at this phenomenon compare with other ways to look at it? What problems do you see in our community that we might analyze? Teachers who relate aspects of mathematics to students' cultures often achieve more equitable outcomes (Hammond, 2014).

Focus

I didn't want to just know the names of things. I remember really wanting to know how it all worked.

—Elizabeth Blackburn, Winner of the 2009 Nobel Prize for Physiology or Medicine.

The principle of *focus* is closely tied to the goal of *depth* of understanding. The principle derives from a need to confront the mile-wide but inch-deep mathematics curriculum experienced by many.

Instructional design built on moving from one content standard to the next underscores the challenging reality that the Standards simply contain *too many* concepts and strategies to address comprehensively in this manner. Teachers often opt to choose between covering standards at an adequate depth (while skipping some topics), or including all topics from the Standards for their grade level and compromising opportunities to reach rich, deep understandings.

One common approach to the coverage-versus-depth challenge is to designate some content standards more important than others. An unintentional result of this, in many schools, is that the standards deemed “less important” simply are not addressed.

The Standards, however, are *not* a design for instruction, and should not be used as such. The Standards lay out expected mastery of content at the grade levels, and expected mathematical practices at the conclusion of high school. They say little about how to achieve that mastery or build those practices.

This framework’s answer to the coverage-vs-depth challenge posed by the principle of *focus* is to lay out principles for (and examples of) instructional design that make the Standards achievable. These principles include as follows:

- Focus on investigations and connections, not individual standards: class activities should be designed around big ideas, and typically should necessitate several clusters of content standards and multiple practice standards, as part of an investigation. Connections between those content standards then becomes an integral part of the class activity, and not an additional topic to cover. The twin focus on investigations and connections is reflected in titles and structure of the grade-banded chapters, Chapters 6, 7, and 8, as well as in the DIs and CCs (see below).
- Tasks must be worthy of student engagement.
    - Problems (tasks which students do not already have the tools to solve) *precede* teaching of the focal mathematics which are necessitated by the problem. That is, the major point of a problem is to raise questions that can be answered, and promote students using their intuition, before learning new mathematical ideas (Deslauriers, McCarty, Miller, Callaghan, & Kestin, 2019).
    - Exercises (tasks which students already have the tools to solve) should either be embedded in a larger context which is motivating (such as the Drivers of Investigations, or exploration of patterns, or games), or should address strategies whose improvement will help students accomplish some motivating goal.

- Students should learn to see their goal as investigating mathematical ideas, asking important questions, making conjectures and developing curiosity about mathematics and mathematical connections.

Rigor

True rigor is productive, being distinguished in this from another rigor which is purely formal and tiresome, casting a shadow over the problems it touches.

—Émile Picard (1905)

In this framework, *rigor* refers to an integrated way in which conceptual understanding, strategies for problem-solving and computation, and applications are learned, so that each supports the other. This definition is more specific and somewhat more demanding than the CA CCSSM's requirement that "*rigor* requires that conceptual understanding, procedural skill and fluency, and application be approached with equal intensity" (CA CCSSM, 2013, p. 2).

This definition expresses the basis of mathematical rigor: reasoning which enables understanding "all the way down to the bottom" (Ellenberg, 2014, p. 48), often expressed in terms of validity and soundness of arguments. According to the definition used here, conceptual understanding cannot be considered rigorous if it cannot be *used* to analyze a novel situation encountered in the world; computational speed and accuracy cannot be called rigorous unless it is accompanied by conceptual understanding of the strategy being used, including why it is appropriate in a given situation; and a correct answer to an application problem is not rigorous if the solver cannot explain to the client both the ideas of the model used and the methods of calculation.

In particular, rigor is *not* about abstraction. In fact, a push for premature abstraction leads, for many students, to an absence of rigor in the sense used in this framework. It is true that more advanced mathematics often occurs in more abstract contexts. This leads many to value more abstract subject matter as a marker of rigor. "Abstraction" in this case usually means "less connected to reality."

But mathematical abstraction is in fact *deeply* connected to reality: When second graders use a representation with blocks to argue that the sum of two odd numbers is even, in a way that other students can see would work for *any* two odd numbers (a representation-based proof; see Schifter, 2010), they have *abstracted* the idea of odd number, and they know that what they say about an odd number applies to one, three, five, etc. (Such an argument reflects SMP.7: Look for and make use of structure.)

Abstraction must grow out of experiences in which students experience the same mathematical ideas and representations showing up and being useful in different contexts. When students figure out the size of a population, after 50 months, with a growth of three percent a month; their bank balance after 50 years if they can earn 3 percent interest per year; and the number of people after 50 days who have contracted a disease that is spreading at 3 percent per day, they will abstract the notion of a quantity growing by a certain percentage per time period, and recognize that they can use the same reasoning in each case to understand the changing quantity.

Thus, the challenge posed by the principle of *rigor* is to provide all students with experiences that interweave concepts, problem-solving (including appropriate computation), and application, such that each supports the other. To meet this challenge, the *Math Framework* emphasizes these principles for designing instruction:

- Abstract formulations should *follow* experiences with multiple contexts that call forth similar mathematical models.
- Contexts for problem-solving should be chosen to provide representations for important concepts, so that students may later use those contexts to reason about the mathematical concepts raised. The Drivers of Investigation (see below) provide broad reasons to think rigorously (“all the way to the bottom”) in ways that linkages between and through topics (Content Connections, see below) are recognized, valued and internalized.
- Computation should serve a genuine need for students to know, typically in a problem-solving or application context.
- Applications should be authentic to students and should be enacted in a way that requires students to explain or present solution paths and alternate ideas.

Assessing for Coherence, Focus and Rigor

In order to gauge what students know and can do in mathematics, we need to broaden assessment beyond narrow tests of procedural knowledge to better capture the connections between content and SMPs. For example, assessing a good mathematical explanation includes how students mathematize a problem, connect the mathematics to the context, and explain their thinking in a clear, logical manner that leads to a conclusion or solution (Callahan, 2020). Helpful math guidelines from the English Learner Success Forum (ELSF) center on focus area five, assessment of mathematical content, practices, and language. Specifically, these guidelines note the need to capture and measure students' progress over time (ELSF guideline 14), and attend to student language produced (ELSF guideline 15).

Designing for Coherence, Focus and Rigor: Drivers of Investigation and Content Connections

With motivating students to learn coherent, focused, and rigorous mathematics as the goal, this framework identifies three **Drivers of Investigation** (DIs), which provide the "why" of learning mathematics, to pair with four categories of **Content Connections** (CCs), which provide the "how and what" mathematics (CA CCSSM) is to be learned in an activity. Together with the Standards for Mathematical Practice, the Drivers of Investigation are meant to propel the learning of the ideas and actions framed in the Content Connections in ways that are coherent, focused, and rigorous.

The following diagram is meant to illustrate the ways that the Drivers of Investigation relate to the Content Connections and Standards for Mathematical Practice. Note that any Driver of Investigation can go with any of the Content Connections and any of the Standards for Mathematical Practice.

Figure 1: Content Connections, Mathematical Practices and Drivers of Investigation

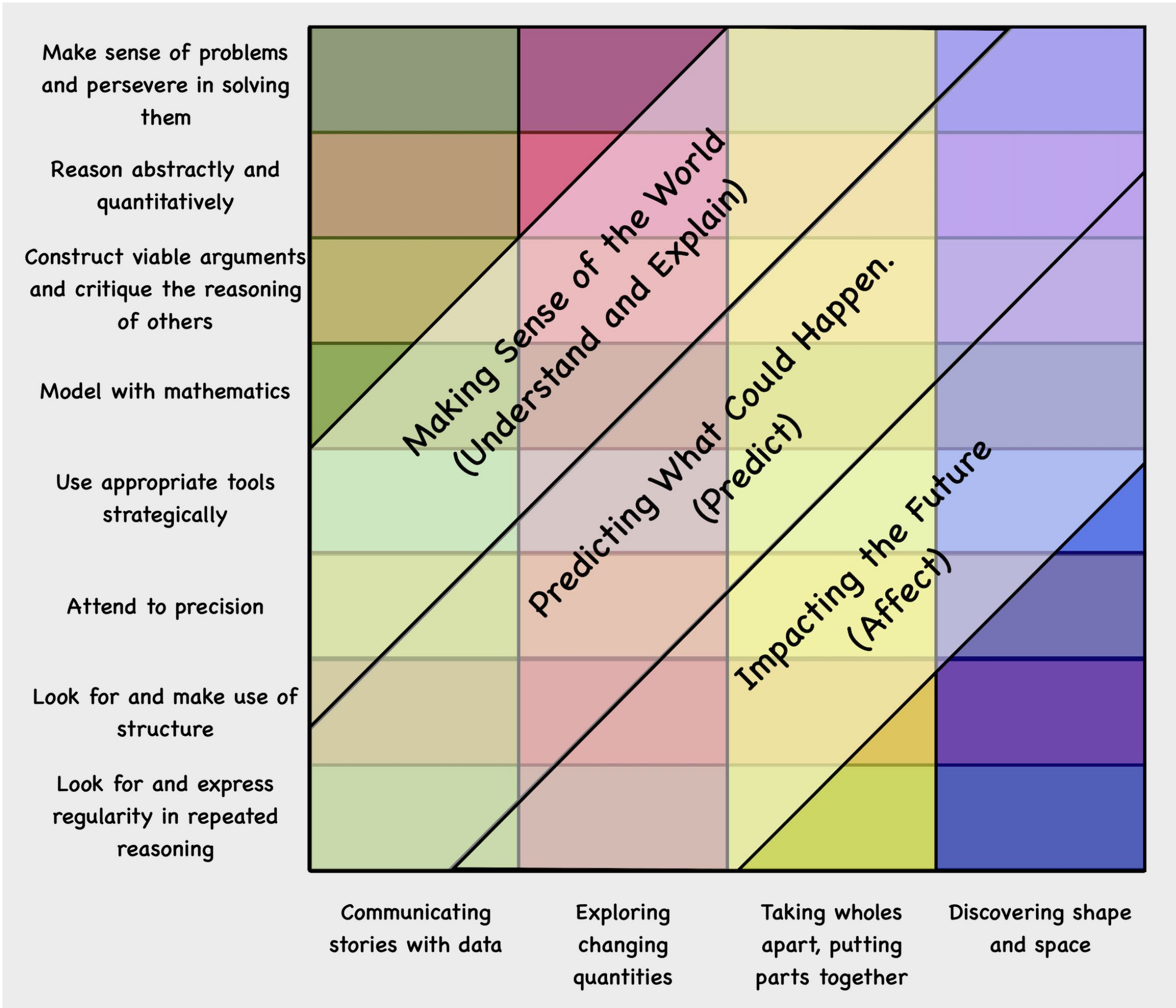

Image long description: Three Drivers of Investigation (DIs) provide the "why" of learning mathematics: Making Sense of the World (Understand and Explain); Predicting What Could Happen (Predict); Impacting the Future (Affect). The DIs overlay and pair with four categories of Content Connections (CCs), which provide the "how and what" mathematics (CA-CCSSM) is to be learned in an activity: Communicating stories with data; Exploring changing quantities; Taking wholes apart, putting parts together; Discovering shape and space. The DIs work with the Standards for Mathematical Practice to propel the learning of the ideas and actions framed in the CCs in ways that are coherent, focused, and rigorous. The Standards for Mathematical Practice are: Make sense of problems and persevere in solving them; Reason abstractly and quantitatively; Construct viable arguments and critique the reasoning of others; Model with mathematics; Use appropriate tools strategically; Attend to precision; Look for and make use of structure; Look for and express regularity in repeated reasoning.

Drivers of Investigation

The Content Connections should be developed through investigation of questions in authentic contexts; these investigations will naturally fall into one or more of the following Drivers of Investigation. The DIs are meant to serve a purpose similar to that of the Crosscutting Concepts in the California Next Generation Science Standards, as unifying reasons that both elicit curiosity and provide the motivation for deeply engaging with authentic mathematics. The aim of the Drivers of Investigation is to ensure that there is always a reason to care about mathematical work, and that investigations allow students to make sense, predict, and/or affect the world. The DIs are:

- DI1: Making Sense of the World (Understand and Explain)
- DI2: Predicting What Could Happen (Predict)
- DI3: Impacting the Future (Affect)

Used in conjunction with the Content Connections, and the Standards for Mathematical Practice, the Drivers of Investigation can guide instructional design. For example, students can make sense of the world (DI1) by exploring changing quantities (CC2) through classroom discussions wherein students have opportunities to construct viable arguments and critique the reasoning of others (SMP.3).

Teachers can use the DIs to frame questions or activities at the outset for the class period, the week, or longer; or refer to these in the middle of an investigation (perhaps in response to the “Why are we doing this again?”-type questions students often ask), or circle back to these at the conclusion of an activity to help students see “why it all matters.” Their purpose is to leverage students’ innate wonder about the world, the future of the world, and their role in that future, in order to motivate productive inclinations (the SMPs) that foster deeper understandings of fundamental ideas (the CCs and the Standards), and to develop the perspective that mathematics is a lively, flexible endeavor by which we can appreciate and understand so much of the inner workings of our world.

Content Connections

The four Content Connections described in the framework organize content and provide mathematical coherence through the grades:

- CC1: Communicating Stories with Data
- CC2: Exploring Changing Quantities
- CC3: Taking Wholes Apart, Putting Parts Together
- CC4: Discovering Shape and Space

Content Connection 1: Communicating Stories with Data

With data all around us, even the youngest learners make sense of the world through data—including data about measurable attributes. In grades TK–5, students describe and compare measurable attributes, classify objects and count the number of objects in each category. In grades 6–8, prominence is given to statistical understanding, reasoning with and about data, reflecting the growing importance of data as the source of most mathematical situations that students will encounter in their lives. In grades 9–12, reasoning about and with data is emphasized, reflecting the growing importance of data as the source of most mathematical situations that students will encounter in their lives. Investigations in a data-driven context—data either generated/collected by students, or accessed from publicly-available sources—help maintain and build the integration of mathematics with students' lives (and with other disciplines such as science and social studies). Most investigations in this category also involve aspects of CC2: *Exploring Changing Quantities*.

Content Connection 2: Exploring Changing Quantities

Young learners' explorations of changing quantities support their development of meaning for operations, and types of numbers. The understanding of fractions established in TK–5 provides them with the foundation they need to explore ratios, rates, and percents in grades 6–8. In grades 9–12, students make sense of, keep track of, and connect a wide range of quantities, and find ways to represent the relationships between these quantities in order to make sense of and model complex situations.

Content Connection 3: Taking Wholes Apart, Putting Parts Together

Students engage in many experiences with taking apart quantities and putting parts together strategically, including utilizing place value in performing operations (such as making 10), decomposing shapes into simpler shapes and vice versa, and relying upon unit fractions as the building blocks of whole and mixed numbers. This Content Connection also serves as a vehicle for student exploration of larger-scale problems and projects, many of which will intersect with other CCs as well. Investigations in this CC will require students to decompose challenges into manageable pieces, and assemble understanding of smaller parts into understanding of a larger whole.

Content Connection 4: Discovering Shape and Space

In the early grades, students learn to describe their world using geometric ideas (e.g., shape, orientation, spatial relations). They use basic shapes and spatial reasoning to model objects in their environment and to construct more complex shapes, thus setting the stage for measurement and initial understanding of properties such as congruence and symmetry. Shape and space work in grades 6–8 is largely about connecting foundational ideas of area, perimeter, angles, and volume notions to each other, to students' lives, and to other areas of mathematics, such as nets and surface area or two-dimensional shapes to coordinate geometry. In grades 9–12, the CA CCSSM supports visual thinking by defining congruence and similarity in terms of dilations and rigid motions of the plane, and through its emphasis on physical models, transparencies, and geometry software.

New to this Framework

To address the needs of California educators in 2021, the *Math Framework* includes several new emphases and types of chapters. Unlike 2013, when the framework featured two separate chapters—one on instruction and one on access—the 2021 framework offers a single chapter, Chapter 2: Teaching for Equity and Engagement, which promotes instruction that fosters equitable learning experiences for all, and challenges the deeply-entrenched policies and practices that lead to inequitable outcomes. While some people argue for a false dichotomy between equity and high achievement, this framework rejects that notion in favor of emphasizing ways good teaching leads to equitable and higher outcomes. Instruction and equity together create instructional designs that can bring about equitable outcomes. The State-level commitment to equity extends throughout the framework, and every chapter highlights considerations and approaches designed to help mathematics educators create and maintain equitable opportunities for all.

Two chapters are devoted to exploring the development, across the TK–12 grade timeframe, of particular content areas. One such area is number sense across TK–12 (Chapter 3: Number Sense), a crucial foundation for all later mathematics and early predictor of mathematical perseverance. The other is data science (Chapter 5: Data Science), which has become tremendously important in the field since the last framework. The other new chapter, Chapter 4: Exploring, Discovering, and Reasoning With and About Mathematics, presents the development of a related cluster of SMPs across the entire TK–12 timeframe. While it is beyond the scope of the *Math Framework* to develop such a “progression” for all SMPs, this chapter can guide the careful work that is required to develop SMPs across the grades. The idea of learning progressions across multiple grade levels is emphasized further in the grade-banded chapters, Chapter 6: Grades TK–5, Chapter 7: Grades 6–8, and Chapter 8: Grades 9–12. The big ideas for each grade band, in the form of overarching Drivers of Investigation and Content Connections, provide a structure for promoting relevant and authentic activities for students, sample tasks, snapshots, and vignettes to illustrate the building of ideas across grades. Chapter 9: Supporting Equitable and Engaging Mathematics Instruction, presents guidance designed to build an effective system of support for teachers as they facilitate learning for their students; it includes advice for administrators and leaders and sets out models for effective teacher learning. Chapter 10: Technology and Distance Learning in the Teaching of Mathematics, describes the purpose of technology in the learning of mathematics, introduces overarching principles meant to guide such technology use, and general guidance for distance learning. Chapter 11: Assessment in the 21st Century, addresses the need to broaden assessment practices beyond answer finding to record student thinking, and to create assessment systems that emphasize growth of leaning over performance. The chapter reviews “Assessment for Learning” and concludes with a brief overview of the Common Core-aligned standardized assessment used in California: the California Assessment of Student Performance and Progress. Chapter 12: Instructional Materials, is intended to support publishers and developers of instructional materials to serve California's diverse student population. This chapter provides guidance for local districts on the adoption of instructional materials for students in grades 9–12, the social content review process, supplemental instructional materials, and accessible instructional materials.

Explicit Focus on Environmental Principles and Concepts. While the Drivers of Investigations and Content Connections are fundamental to the design and implementation of this framework and the standards, teachers must be mindful of other considerations that are a high priority for California's education system including the Environmental Principles and Concepts (EP&Cs) which allow students to examine issues of environmental and social justice.

References

Abiola, O., & Dhindsa, H. S. (2011). Improving classroom practices using our knowledge of how the brain works. *International Journal of Environmental & Science Education, 7*(1), 71–81.

Aguirre, J., Mayfield-Ingram, K., & Martin, D. (2013). The impact of identity in K-8 mathematics: Rethinking equity-based practices. The National Council of Teachers of Mathematics.

Anderson, R; Boaler, J.; Dieckmann, J. (2018) Achieving Elusive Teacher Change through Challenging Myths about Learning: A Blended Approach. Education Sciences, 8 (3), 98.

Blackwell, L.S; Trzesniewski, V. & Dweck, C. S. "Implicit Theories of Intelligence Predict Achievement Across an Adolescent Transition: A Longitudinal Study and an Intervention," *Child Development* 78/1 (2007): 246–63.

Boaler, Jo.; Schoenfeld, A.; Daro, P.; Asturias, H.; Callhan, P. & Foster, D. (2018). How One City Got Math Right: Pathways that work. The Hechinger Report, October 9, 2018. https://hechingerreport.org/opinion-how-one-city-got-math-right/.

Boaler (2019a). *Limitless Mind. Learn, Lead and Live without Barriers*. Harper Collins.

Boaler, J. (2019b). Prove it to Me! Mathematics Teaching in the Middle School, 422–429.

Boaler, J. (2019c). Everyone Can Learn Mathematics to High Levels: The Evidence from Neuroscience that Should Change our Teaching. American Mathematics Society. https://blogs.ams.org/matheducation/2019/02/01/everyone-can-learn-mathematics-to-high-levels-the-evidence-from-neuroscience-that-should-change-our-teaching/#more-2392.

Boaler, J., Cordero, M., & Dieckmann, J. (2019). Pursuing Gender Equity in Mathematics Competitions. A Case of Mathematical Freedom. Mathematics Association of America, *FOCUS*, Feb/March 2019. http://digitaleditions.walsworthprintgroup.com/publication/?m=7656&l=1#{%22issue_id%22:566588,%22page%22:18}.

Boaler, J., Chen, L., Williams, C., & Cordero, M. (2016). Seeing as Understanding: The Importance of Visual Mathematics for our Brain and Learning. Journal of Applied & Computational Mathematics, 5(5), DOI: 10.4172/2168-9679.1000325.

Boaler, J.; Dieckmann J.; Pérez-Núñez G.; Sun K., and Williams C. (2018) Changing Students Minds and Achievement in Mathematics: The Impact of a Free Online Student Course. *Frontiers*. Educ. 3:26.

Boaler, J. & LaMar, T. (2019). There is a Better Way to Teach Students with Learning Disabilities. TIME Magazine. https://time.com/5539300/learning-disabilities-special-education-math-teachers-parents-students/.

Bressoud, D. (Ed.). (2017). The role of calculus in the transition from high school to college mathematics. Retrieved from Mathematical Association of America website: https://www.maa.org/sites/default/files/RoleOfCalc_rev.pdf.

Bressoud, David (2012). MAA/NCTM Joint Position in Calculus. http://launchings.blogspot.com/2012/04/maanctm-joint-position-on-calculus.html.

Bressoud, D. M., Mesa, V., & Rasmussen, C. L. (Eds.). (2015). *Insights and recommendations from the MAA national study of college calculus*. MAA Press.

Butler, R. (2008). Ego-involving and frame of reference effects of tracking on elementary school students' motivational orientations and help seeking in math class. *Social Psychology of Education*, *11*(1), 5–23.

Carr, P.B., Dweck, C.S. & Pauker, K. (2012) "'Prejudiced' Behavior Without Prejudice? Beliefs About the Malleability of Prejudice Affect Interracial Interactions," *Journal of Personality and Social Psychology* 103/3 (2012): 452.

Charles, R. (2005) "Big Ideas and Understandings as the Foundation for Elementary and Middle School Mathematics," *Journal of Mathematics Education Leadership,* 7(3), 9–24.

Coyle, D. (2009). *The Talent Code: Greatness Isn't Born. It's Grown. Here's How.* (New York: Bantam).

Devlin, K. J. (2000). *The math gene: How mathematical thinking evolved and why numbers are like gossip*. New York: Basic Books.

Doidge, N. (2007). The brain that changes itself: Stories of personal triumph from the frontiers of brain science. Penguin.

Ellenberg, J. (2015). How not to be wrong: The power of mathematical thinking. Penguin.

Feigenson, L., Dehaene, S., & Spelke, E. (2004). Core systems of number. *Trends in cognitive sciences*, *8*(7), 307–314

González, N., Moll, L. C., & Amanti, C. (Eds.). (2006). *Funds of knowledge: Theorizing practices in households, communities, and classrooms*. Routledge.

Heller, E. (1952). *The disinherited mind: Essays in modern German literature and thought*. Cambridge: Bowes & Bowes.

Huber, E., Donnelly, P.M., Rokem, A. *et al.* Rapid and widespread white matter plasticity during an intensive reading intervention. *Nat Commun* 9, 2260 (2018). https://doi.org/10.1038/s41467-018-04627-5.

Joseph, N. M., Hailu, M., & Boston, D. (2017). Black women's and girls' persistence in the P–20 mathematics pipeline: Two decades of children, youth, and adult education research. Review of Research in Education, 41(1), 203–227.

Lambert, R. (2018). "Indefensible, illogical, and unsupported"; countering deficit mythologies about the potential of students with learning disabilities in mathematics. *Education Sciences*, *8*(2), 72.

Langer-Osuna, J. M. (2011). How Brianna became bossy and Kofi came out smart: Understanding the trajectories of identity and engagement for two group leaders in a project-based mathematics classroom. Canadian Journal of Science, Mathematics and Technology Education, 11(3), 207–225.

Lockhart, P. (2009). *A mathematician's lament: How school cheats us out of our most fascinating and imaginative art form*. Bellevue literary press. Available online at https://www.maa.org/external_archive/devlin/LockhartsLament.pdf.

Leslie, S. J., Cimpian, A., Meyer, M., & Freeland, E. (2015). Expectations of brilliance underlie gender distributions across academic disciplines. *Science*, *347*(6219), 262–265.

Letchford, L. (2018). Reversed: A memoir. Acorn Publishing.

MAC & CAASPP (2015) Technical Report, Years 2014 and 2015, Educational Data Systems, 15850 Concord Circle, Morgan Hill.

Maguire, E. A., Woollett, K., & Spiers, H. J. (2006). London taxi drivers and bus drivers: A structural MRI and neuropsychological analysis. Hippocampus, 16(12), 1091–1101

Menon, V. (2015). "Salience Network," in Arthur W. Toga, ed., *Brain Mapping: An Encyclopedic Reference*, vol. 2 (London: Academic), 597–611.

Moschkovich, J. (2013). Principles and guidelines for equitable mathematics teaching practices and materials for English language learners. *Journal of Urban Mathematics Education*, *6*(1), 45–57.

Moser, J.S. et al. (2011). “Mind Your Errors: Evidence for a Neural Mechanism Linking Growth Mind-set to Adaptive Posterror Adjustments,” *Psychological Science* 22/12 (2011): 1484–89.

National Council of Supervisors of Mathematics (NCSM) & TODOS: Mathematics for ALL. (2016). Mathematics education through the lens of social justice: Acknowledgement, actions, and accountability.

Park, J., & Brannon, E. (2013). Training the approximate number system improves math proficiency. *Association for Psychological Science*, 1–7.

Parker, P. D., Marsh, H. W., Ciarrochi, J., Marshall, S., & Abduljabbar, A. S. (2014). Juxtaposing math self-efficacy and self-concept as predictors of long-term achievement outcomes. *Educational Psychology*, *34*(1), 29–48.

Philipp, R. A. (2007). Mathematics teachers’ beliefs and affect. *Second handbook of research on mathematics teaching and learning*, *1*, 257–315.

Piaget, J., & Cook, M. (1952). *The origins of intelligence in children* (Vol. 8, No. 5, p. 18). New York: International Universities Press.

Picard, E. (1905). On the development of mathematical analysis and its relation to certain other sciences. *Bulletin of the American Mathematical Society*, *11*(8), 404–426.

Schifter, D. (2010). Representation-based proof in the elementary grades. In *Teaching and learning proof across the grades* (pp. 71–86). Routledge.

Schoenfeld, A. H. (2014). What makes for powerful classrooms, and how can we support teachers in creating them? A story of research and practice, productively intertwined. *Educational researcher*, *43*(8), 404–412.

Student Achievement Partners (undated). Mathematics: Focus by Grade Level. https://achievethecore.org/category/774/mathematics-focus-by-grade-level.

Su, F. (2020). *Mathematics for human flourishing*. Yale University Press.

Turner, E. E., & Celedón-Pattichis, S. (2011). Mathematical problem solving among Latina/o kindergartners: An analysis of opportunities to learn. *Journal of Latinos and Education*, *10*(2), 146–169.

Williams, T. (2018). *Power in numbers: The rebel women of mathematics*. Race Point Publishing.

Woollett, K., & Maguire, E. A. (2011). Acquiring “the Knowledge” of London’s layout drives structural brain changes. Current Biology, 21(24), 2109–2114.